Archimedean, Density & Inequality

Theorem (cf. Th 2.4.3 in Bartle) **This theorem has six parts of which (I) and (II) are usually referred as Archimedean Property. Proof is given immediately after the statement of each part.**

(I) Let x be a real number than there exists a natural number n >x.

Proof. If not then x is an upper bound of the set **N** of natural numbers and hence, by the Axiom III, sup (N) exists in R: - let it be denoted by u : =sup (**N**). Note that u-1 < u so u-1 is NOT an upper bound of N and so u-1 < n for some natural number n and hence u < 1+n and so

$$(10+n) \leq u < 1+n \qquad \text{(absurdity)}.$$

(u being an upper bound of $\mathbb{N}$ & $(10+n) \in \mathbb{N}$).

(II). Let t >0. Then there exists a natural number n such that 1/n < t.

Proof. Applying (I) to 1/t in place of x, take a natural number n such that 1/t < n (so 1/n < t because n and t are positive).

(III). Let x > 1. Then there exists (uniquely) a natural number n (usually denoted by [x]) such that

$$n \leq x < n+1 \qquad (*)$$

Proof. By the well-order principle, there exists the largest natural number n dominated by or equal to x. Equivalently the above displayed inequalities (*) hold.

(IV) Let x be a real number. Then there exists (uniquely) an integer n satisfying (*)

Proof. Extend the well-order principle to Z (the set of integers : If Y is a nonempty subset of Z and is bounded above then Y has the largest element.

(V) Density of Q (the set of rational numbers). Let real numbers x < y. Then there exists a rational number r such that x < r <y.

Proof. Progressively we consider the cases below.

(1) Suppose 1 < x < y and y - x > 1. Then the integral part [x] of x satisfies

$$[x] \in \mathbb{N} \subseteq \mathbb{Q}$$

$$[x] \leq x < [x]+1 < y,$$

(the last inequality holds thanks to the first inequality and the assumption that y > x +1.)

Thus **[x] + 1** has the property required for r.

(2) Suppose 1 < x < y. Then, by the Archimedean Property (**II,** Applied to the positive number y - x), there exists a natural number m such that (1/m) < y-x. Then my -mx > 1 and it follows from case (1) (applied to mx, my in place of x, y) that there exists a natural number n such that mx < n <my, and so n/m is a rational number lying between x, y.

(3) The general case: x <y. By the Archimedean Property **I**, take a natural number k such that k > -x and so -k <x <y and 1 < x+k+1 <y+k+1. By (II), there exists a rational r lying between x+k+1 and y+k+1 and so r-(k+1) is a rational lying between x and y.

Exercise

1. Let x< y. Then there exist natural numbers m, n such that x+ 1/m < y-1/n.
Hint: Take n such that 1/n < y-x and then 1/m < y-x-(1/n). Or simply take m = n < (y-x)/2.

2. Let a, b be positive numbers. The a < b iff a^2 < b^2 (iff 0 < b^2 -a^2 = (b-a) (b+a) iff 0 < (b-a) because b+a and (b+a)^-1 are positive).

3. Let x, y be positive real numbers such that x^2 < a and y^ 2 > b . Show that there exist natural numbers m, n such that (x + 1/n)^2 < a and (y -1/m)^2 > b.
Hint: The first requirement is x^2 + 2x/n+ 1/(n)^2 < a which would be satisfied if x^2 +2x/n + 1/n <a as 1/(n^2) is smaller (or equal to) 1/n. Such natural number n does exist by Archimedean property **II**. Similarly for the 2nd part of this exercise.

(VI) Square Root and Density of Irrationals. There exists (unique) z > 0 such that z^2 = 2 (that is, z is the positive sq root of 2). R\Q is dense : if x < y then there exists an irrational number t such that x < t < y

Proof. Let A = { a : 0 < a and a^2 < 2}, e.g., 1 belongs to A but A is bounded above by 2 because a^2 < 2 <2^2 and so a < 2 for all a in A. By Axiom III, let z: =sup A. Then z lies in [1, 2]. Shall show that z^2 = 2 by showing that z^2 cannot be bigger nor smaller than 2 as detailed below.

Suppose z^2 < 2. Then, by ....., there exists a natural number n such that (z +1/n)^2 < 2 and so (z+1/n) belongs to A and so is dominated by z (which is not possible as 1/n is positive), being the supremun of A.

Next consider the case z^2 > 2. Then, by ....., there exists a natural number m such that (z -1/m)^2 > 2 > a^2 for all a in A and so z-1/m > a for all a in A. This implies that z - 1/m dominates z by definition of z; again this is absurd as -1/m is negative. This completes the proof for the first part of **(VI).** For the 2nd part, let x < y and take (Why exists?) a rational r such that x < r <y and then (with z = sq roof of 2) take a natural number n such that r + z/n < y. Then r+z/n is an irrational number lying between x and y.

Lemma on Inequality ('Making life easier' Lemma)

Suppose $x \leq y+\varepsilon$ (or $x < y+\varepsilon$) for all $\varepsilon > 0$. Then $x \leq y$. In particular if $|x| \leq \varepsilon$ $\forall \varepsilon > 0$ then $x = 0$.

Proof. Suppose not : $x > y$. Let $\varepsilon = \frac{x-y}{2}$. Then $\varepsilon > 0$ but

$$y+\varepsilon < y + (x-y) = x,$$

contradicting the assumption that $y+\varepsilon \geq x$. This proves the first assertion. The 2nd follows immediately.

Absolute Values & Triangle Inequality.

$$|x| := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases} \qquad (x \in \mathbb{R})$$

i.e. $|x| = \begin{cases} x & \forall x \geq 0 \\ -x & \forall x \leq 0 \end{cases}$

Thus $0 \leq |x| = x$ or $-x \quad \forall x \in \mathbb{R}$

& $\quad |x| < r \iff -r < x < r \quad (\forall\, r > 0)$

i.e. $\pm x < r$

Proposition. Let $x, y, z \in \mathbb{R}$. Then

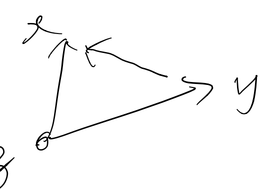
(i) $|-z| = |z|$ (regardless $z \geq 0$ or $z < 0$)

more generally $|cz| = |c||z| \ \forall\, c, z \in \mathbb{R}$

(ii) $|x+y| \leq |x| + |y|$ $\left(\& \ |x-y| \leq |x| + |y| \right)$

(iii) $|x-z| \leq |x-y| + |y-z|$

(iv) $\big| |x| - |y| \big| \leq |x-y|$ $\left(\because \pm(|x| - |y|) \leq |x-y| \right)$



proof. $x + y \leq |x| + |y|$

& $-(x+y) \leq |x| + |y|$ so (ii) holds.

(iii) follows easily from (ii).

(iv) By (iii) and symmetry,

$|x| - |y| \leq |x-y| \quad \& \quad |y| - |x| \leq |x-y|$

so (iv) follows.

Ex1. Let $a, b \in \mathbb{R}$. Then

$$\max\{a,b\} = \frac{a+b+|a-b|}{2} \quad \& \quad \min\{a,b\} = \frac{a+b-|a-b|}{2}$$

i.e. what you learnt in Primary School

"大数" $= \frac{\text{和}+\text{差}}{2}$ & "细数" $= \frac{\text{和}-\text{差}}{2}$

Ex 2 (Binomial Theorem) $(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + nab^{n-1} + b^n$

In particular $(1+b)^n = 1 + nb + \frac{n(n-1)}{2!} b^2 + \frac{n(n-1)(n-2)}{3!} b^3$

$+ \dots \frac{n(n-1)\cdots 3.2}{(n-1)!} b^{n-1} + b^n$

Noting each term in the expansion is positive so

$(1+b)^n \geq 1 + nb$ etc. provided $b > 0$ Show

Ex 3. Bernoulli inequality

$(1+b)^n \geq 1+nb \quad \forall\ b > -1$

Hint on Ex 3. Use MI (not Binomial Expansion)